Learning, Mean Field Approximations, and Phase Transitions in Auction Models.

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Abstract

In this paper we propose a learning model for bidding in multi-round, pay as bid, sealed bid auctions using techniques from partial differential equations and statistical mechanics tools. As an application, we perform a theoretical study of an agent based model.

We assume that in each round a fixed fraction of bidders is awarded, and bidders learn from round to round using simple microscopic rules, adjusting myopically their bid according to their performance. Agent-based simulations show that bidders coordinate in the sense that they tend to bid the same value in the long-time limit. Moreover, this common value is the true cost or the ceiling price of the auction (for reverse auctions), depending on the level of competition. A discontinuous phase transition occurs when half of the bidders win.

The purpose of this paper is to introduce this theoretical methodology, and to analyze the dynamics. After establishing the rate equations, we obtain their continuous limit, which is a first-order, non-linear partial differential equation. We study its solutions, we prove the existence of the phase transition, and we explain the qualitative behavior of the solutions observed in the agent-based simulations.

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1 Introduction

Auction theory is an active branch of game theory that provides us with mechanisms to buy or sell different goods. With a correct design, the goods can be allocated efficiently, since the bidders with higher valuations are more likely to obtain them.

The classical theory comprises many techniques to determine optimal bidding policy in several cases, see for instance [19, 20, 24], helping the auctioneer to estimate the result of the auction process. The radio frequency spectrum and television channel frequency auctions in the 1980s and 1990s (see [25]) were followed by spectrum auctions for different generations of mobile communications [17], and they generate different auction structures. Since few bidders can compete in these very concentrated markets, they posed serious challenges of collusive bidding, and tacit collusion was one of the risks of *learning*.

Meanwhile, computer science entered this world in recent decades, motivated by word search auctions on Google, Facebook, or Amazon, to name a few, and by online auctions at sites including eBay, Alibaba, and Mercado Libre. New, different problems resulted: billions of online auctions are held each day, millions of words or objects are auctioned simultaneously, and thousands of agents interact repeatedly and almost anonymously. Hence, combinatorial considerations make them untractable with classical tools, and many algorithms were developed to manage huge volumes of data, trying to understand how to bid, see for instance the recent review [27] and the references therein.

We can think of classic auctions as a microscopic world, with individualized bidders and goods, while online auctions occur in a high dimensional landscape, a macroscopic world, where only trends, means, and variances are observable. Exact computations are valid only in the first, and heuristic arguments can be used safely in the second without precisely knowing the full details.

However, there is something in between, and some mesoscopic scale is needed. Renewable energy (RE) auctions are very different from auctions selling words or collectible items online, or auctions of telecommunications licenses. Few auctions are held, and data is scarce due to local differences in legislation, taxes, and credit availability. Costs and technologies change rapidly each year, participation is expensive, and large amounts of money are at stake to learn from experience. However, the number of bidders is high enough to prevent exact computations, and there is a high degree of heterogeneity, since project size can range from 0.25 MW/h to 100MW/h; bidders can be local or international developers with high variability in credit access; and solar, wind, geothermal, and other technologies depend heavily on the natural resources. Usually, several rounds are held on a periodic basis, and firms clearly adapt their behavior, reacting to other players bids.

Nowadays, RE auctions are analyzed through agent based simulations [4, 16, 22, 23]. We recently proposed in [32] a model of such auctions taking place in several rounds where participants can adapt their strategy from round to round. We considered a myopic learning rule where each bidder updates its strategy considering only their performance in the round without considering its history, nor that of others bidders. This leads to a simple model much in the spirit of the ones introduced by physicists to study social and economic behaviors, see [8, 10, 34]. Despite its simplicity, agent-based simulations show that this model can reproduce outcomes of real RE auctions reasonably well.

The main purpose of this paper is to provide a theoretical framework to analyze learning in auctions using ordinary and partial differential equations. We apply it to this model and explain some of the results observed in the agent-based simulations. We first present the model and some simulations in Section §2; briefly, each agent has a parameter μ, and submits a bid drawn from a normal distribution $N(\mu, \sigma)$. We then address in Section §3 the theoretical analysis of the long-time behavior of the distribution of the population in the main parameter $\mu \in [0, 1]$ of the model, where up to translations and normalization, 0 represents the minimal cost, and 1 the ceiling prize. We first discretize $[0, 1]$ with a mesh $\{0, \gamma, 2\gamma, \ldots, 1\}$ of step γ, and we obtain the system of rate equations giving the evolution of the expected proportion of bidders with $\mu = 0, \gamma, 2\gamma, \ldots, 1$. This is a system of approximately $1/\gamma \gg 1$ coupled nonlinear ordinary differential equations which can provide some useful information. To obtain better insight into the dynamic, we derive its continuous limit as $\gamma \to 0$ thus obtaining a first-order, non-local, partial differential equation. This equation is also deduced adopting a kinetic point of view in the spirit of [26]. The solutions of this equation must be understood in a weak sense, in the space of probability measures, which is a very flexible setting where few agents can be considered as sums of Dirac's delta functions.

Let us stress that the use of partial differential equations is quite unusual in the study of auctions. They appeared recently in evolutionary game theory generalizing the replicator systems (see [1, 2, 6, 30, 31]) when players adapt their behavior on mixed strategies. The equation obtained is non-linear and non-local, and thus non-trivial to study. We can nevertheless solve it in some particular cases and then obtain the qualitative behaviour of the solutions, while numerical methods enable us to deal with other cases.

Finally, we prove the existence of a phase transition with respect to the competition level of the auction. It is a well known fact that participation is a key factor in competitive auction. *No amount of bargaining power is as valuable to the seller as attracting one extra bona fide bidder*, as Bulow and Klemperer said in [7]. However, there is a lack of quantitative estimates, and we believe that this technique can be used to obtain them in other settings. The most technical proofs are placed in the Appendix to avoid interrupting the flow of the paper.

2 Model description

We present in this section our model of auctions and some simulations. We consider an auction taking place in several rounds $t = 1, \ldots, T$ involving the same N bidders.

Remember that we are motivated by RE procurement auctions where bidders compete to gain the right to sell energy on a pay as bid basis, and submit their prices, the higher the better for them (but the opposite for the auctioneer). We suppose that whatever the round, each bidder always offer a fixed volume v of energy, and the auctioneer always auctions the same volume V of energy where V is supposed to be a multiple of v. The ratio $N_w = V/v$ is thus the number of winners in each round. We let $p = N_w/N$ the proportion of winners, and call $\rho = 1/p$ the competition level.

In real auctions the auctioneer can publicly set a ceiling price, that is a maximum accepted bid price so that any higher bid is automatically disqualified. Up to a rescaling we can assume that the ceiling price is 1, and that bidders know the ceiling price.

In each round, each bidder $i = 1, \ldots, N$ submits its bid b_i drawn independently at random from a normal distribution $N(\mu_i, \sigma^2)$. The noise variance σ^2 is constant in time and identical for all bidders, contrary to the mean bid μ_i which will change from round to round as bidders learn. We suppose that bidders all have the same minimum bid value which we can assume without loss of generality to be 0. Recalling that the ceiling price is 1, we thus truncate each bid b_i to $[0, 1]$.

The auctioneer then sorts the bids in ascending order and the lowest N_w bids win. Based on its result each bidder i updates its μ_i to μ_i' raising it (or respectively lowering it) by a fixed amount γ if he just won (or, respectively, lost):

$$\mu_i' := \begin{cases} \min\{\mu_i + \gamma, 1\} & \text{if bidder } i \text{ won,} \\ \max\{\mu_i - \gamma, 0\} & \text{if bidder } i \text{ lost.} \end{cases} \tag{2.1}$$

Here $\gamma > 0$ is the learning parameter, the same for every bidder and round. Notice that μ_i' is truncated to remain in $[0, 1]$. This rule models a myopic behavior in the sense that bidders increase or decrease their mean bid (i.e., their expected profit in case of winning) only by considering how well they performed in the round.

We can thus summarize the three steps of a round as follows:

Step 1 Bidding process: bidders independently submit their bid $b_i \sim N(\mu_i, \sigma^2)$ truncated to remain in $[0, 1]$.

Step 2 Determination of the winners: the auctioneer sorts the bids $b_1, ..., b_N$ in ascending order. The bidders who submitted the lowest $N_{w,t}$ bids are the winners.

Step 3 Learning: each bidder i updates her parameter μ_i to μ_i' using (2.1).

Notice that in this simplified model of auction, p (or equivalently ρ) is the unique parameter concerning the design of the auction and the bidding behaviour (i.e., the strategy) of bidder i is defined by μ_i. Denote f_t the distribution of the parameter μ in the population at time t. It is a probability measure over $[0, 1]$. Our purpose is thus to study the asymptotic behaviour of f_t when $t \to +\infty$ as a function of the initial distribution f_0 and ρ.

It is convenient from a theoretical point of view to consider $\gamma \ll 1$. Observable effects then only appear when T is large. This is of course unrealistic but it will prove very useful theoretically, especially concerning the impact of the competition level, and will enable us to present the mean field equations of the dynamics. Indeed, larger values of γ can be understood as a perturbation, tracing back their effect through higher order terms in the differential equations.

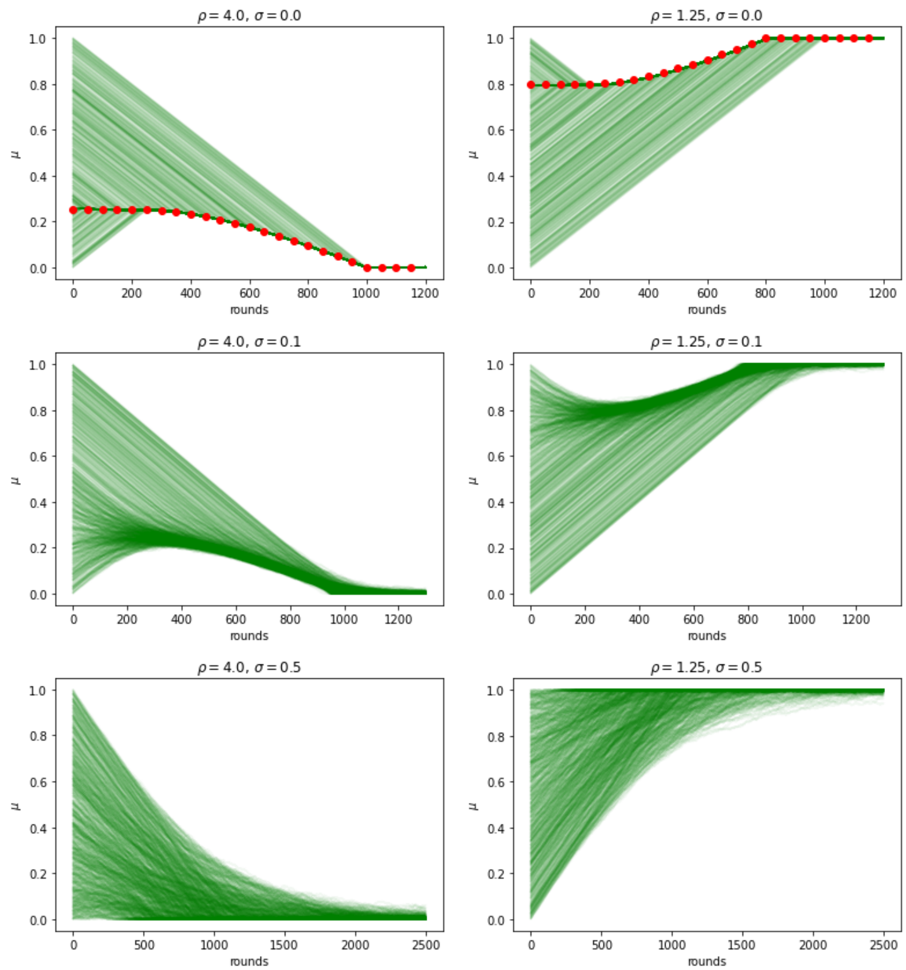

Figure 1: Evolution from round to round of the μ values of bidders (blue curves) for different noise variance σ^2 and competition level ρ (left to right: $\rho = 4$, $\rho = 1.25$, Top to Bottom: $\sigma = 0$, $\sigma = 0.1$, $\sigma = 0.5$). μ's are initially uniformly distributed in $[0, 1]$ and $\gamma = 0.001$. The red points for the simulation with $\sigma = 0$ (top row) indicate the location $a(t)$ of the Dirac mass $\delta_{a(t)}$ as given in Prop. 4 (where time is given in the τ scale, $\tau = \gamma t$).

To gain intuition we first show agent-based simulations of the model. A detailed theoretical study is presented in the next section. We consider $N = 1000$ bidders whose μ parameters are initially drawn at random independently and uniformly in $[0, 1]$ (i.e., f_0 is the probability measure $1_{[0,1]}dx$). We set the learning parameter to $\gamma = 0.001$. We show in Figure 1 the evolution from round-to-round of the μ parameter of the N bidders for two competition level values (left: $\rho = 4$, right: $\rho = 1.25$), and three different values of σ (from top to bottom: $\sigma = 0$, $\sigma = 0.1$, and $\sigma = 0.5$). We can observe that bidders tend to share asymptotically very similar μ values: close to the ceiling price 1 when $\rho = 1.25$, and close to the minimum bid value 0 when $\rho = 4$ (as $\sigma \downarrow 0$, the fluctuations around the limit value appear to go to 0).

Further simulations (not shown here) varying the value of ρ and the initial distribution f_0 of μ's show that bidders always coordinate and the limit common μ value is always close to 0 or 1 depending on the value of $\rho \neq 2$ (equivalently $p \neq 1/2$). We plot in Figure 2 the final common value of μ as a function of the proportion of winners $p = 1/\rho$ (averaged over 10 runs). We can clearly observe a transition at $p = 1/2$ (i.e., $\rho = 2$): when $p > 1/2$ (i.e., $\rho < 2$), the final μ value is close to 1, whereas it is close to 0 for $p < 1/2$ (i.e., $\rho > 2$).

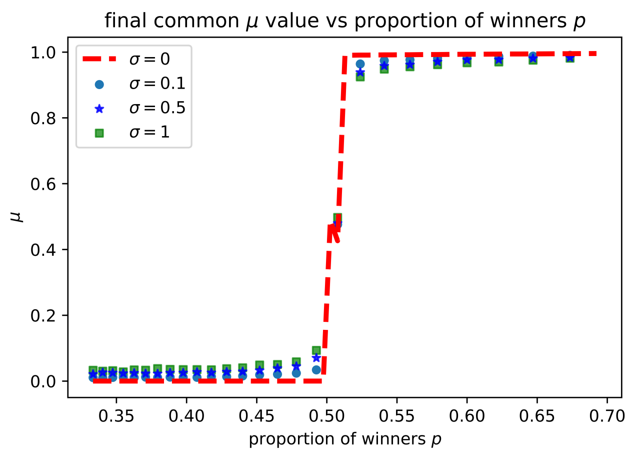

Figure 2: Final common value of μ as a function of proportion of winners $p = 1/\rho$ for different values of noise σ.

In the next section we deduce an equation for the evolution of the distribution f_t of μ's which will help theoretically understand this phase transition and obtain a qualitative picture of the evolution of f_t.

3 Theoretical analysis

In this section we present a theoretical analysis of the dynamic of the model we introduced in the previous section. We want in particular to understand the origin of the transition phase at $\rho = 2$ we observed numerically in Figure 2.

Let us briefly outline the different steps of our analysis. We discretize $[0, 1]$ with mesh size $\gamma \ll 1$ which is consistent with the dynamic since μ's are updated by steps of size γ (see (2.1)). We take γ such that $K = 1/\gamma \in \mathbb{N}$. We can then formulate a system of ordinary differential equations, the rate equations, for the expected proportion of bidders with $\mu = k\gamma$, $k = 0, \dots, K$, see system (3.2) below. This system already provides some insight into the dynamics in some particular cases but becomes intractable when $\gamma \sim 0$ since it consists of $\simeq 1/\gamma$ equations. Passing to the continuum limit taking $\gamma \to 0$ we can approximate it by a first order partial differential equation, see equations (3.5) and (3.6) below. We will also deduce this equation adopting a kinetic point of view. We can then exactly solve this equation when the μ values are initially distributed uniformly in $[0, 1]$ thus recovering the results of the numerical simulations shown in Figure 1. We then analyze the long time behaviour of a solution starting from any initial distribution of μ and prove the occurrence of a transition at $\rho = 2$, thus explaining Figure 2.

3.1 Rate equations.

3.1.1 Deduction of the rate equations

Let us denote by $n_k(t)$ the expected proportion of bidders with $\mu = k\gamma$, for $k = 1, \dots, K$, and let $F_k := n_0 + ... + n_k$ be the cumulative proportion of bidders with $\mu \leq k\gamma$, $k = 0, 1, \dots, K$.

We assume for simplicity that $\sigma = 0$ so that bidders bid exactly their μ. Denoting by $p := 1/\rho$ the proportion of winners, we observe that a bidder with $\mu = k\gamma \in [0, 1]$ surely wins when $F_k \leq p$. When $F_k > p$, the agent may also win when $F_{k-1} < p$. Indeed in that case all bidders with $\mu \leq (k-1)\gamma$ win but they are not enough to reach a percentage p of winners. The remaining $p - F_{k-1}$ percentage of winners is then chosen uniformly among the n_k bidders with $\mu = k\gamma$. In conclusion, a bidder with $\mu = k\gamma$ wins with probability

$$P_k := 1_{F_k \leq p} + \frac{p - F_{k-1}}{n_k} 1_{F_{k-1} < p < F_k} \tag{3.1}$$

(with the convention $F_{-1} := 0$). We thus obtain the following rate equations describing the temporal evolution of $n_0, n_1, ..., n_K$:

$$\begin{aligned} n_0'(t) &= -n_0 P_0 + n_1(1 - P_1), \\ n_k'(t) &= n_{k-1} P_{k-1} + n_{k+1}(1 - P_{k+1}) - n_k \qquad k = 1, ..., K-1, \\ n_K'(t) &= n_{K-1} P_{K-1} - n_K(1 - P_K). \end{aligned} \tag{3.2}$$

These equations follow by noticing that the proportion of bidders with $\mu = k\gamma$ increases due to the inflow of bidders using $\mu + \gamma$ (respectively, $\mu - \gamma$) who lost (or, respectively, won) the round, and decreases due to the outflow of bidders using μ who will end up with $\mu \pm \gamma \neq \mu$.

The next proposition states that this system has a unique global solution for any initial condition (see Appendix for the proof):

Proposition 1. *For any initial condition* $(n_0(0), \ldots, n_K(0))$ *with* $n_k(0) \geq 0$, $k = 0, \ldots, K$, *and* $n_0(0), \ldots, n_K(0) = 1$, *there exists a unique solution* $(n_0(t), \ldots, n_K(t))$ *to the system* (3.2) *which is defined for any* $t \in \mathbb{R}$ *and satisfies* $n_k(t) \geq 0$, $k = 0, \ldots, K$, *and* $n_0(t), \ldots, n_K(t) = 1$ *for any* $t \in \mathbb{R}$.

To verify that the rate equations (3.2) faithfully reproduce the agent-based simulations, we solved numerically (3.2) with $K = 1 + 1/0.001$ ($\gamma = 0.001$) and uniform initial condition ($n_0(0) = 1/(K+1)$, $k = 1 \ldots, K$). We solved the system in Python using the function `scipy.integrate.solve_ivp` and plotted the solution using `matplotlib.pyplot.imshow`. The result is displayed in Figure 3. We can see that the solution of the rate equations system is almost indistinguishable from the agent-based simulations of Figure 1 (top row).

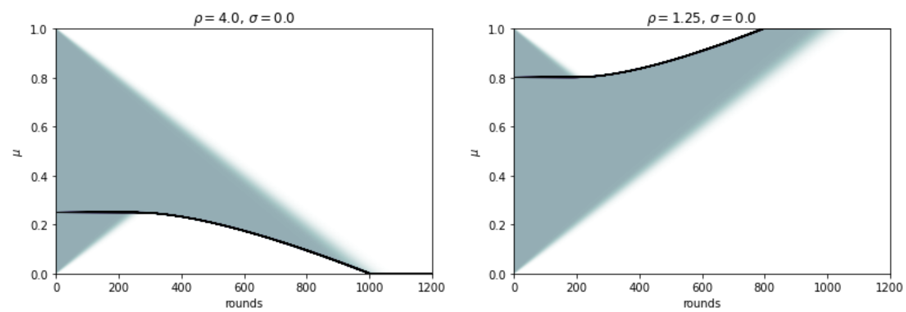

Figure 3: Evolution of the solution $(n_0(t), ..., n_K(t))$ of the rate equations system (3.2) where $K = 1 + 1/0.001$ (which corresponds to $\gamma = 0.001$), with uniform initial distribution.

3.1.2 Simple consequences of the rate equations.

We can already deduce some simple facts from the rate equations (3.2). For instance, when $p = 0$, i.e., there is no winner, then $P_k = 0$, $k = 0, ..., K$, so (3.2) becomes

$$\begin{aligned} n_0'(t) &= n_1, \\ n_k'(t) &= -n_k + n_{k+1} \qquad k = 1, \ldots, K-1, \\ n_K'(t) &= -n_K. \end{aligned}$$

It follows that $n_K, n_{K-1}, ..., n_1 \to 0$ as $t \to +\infty$. Since $n_0 + ... + n_K = 1$, we deduce that $n_0 \to 1$. Thus as $t \to +\infty$, bidders end up bidding $\mu = 0$ (their cost) as expected. Similarly, when $p = 1$, all bidders win, then $P_k = 1$, $k = 0, \ldots, K$, so

$$\begin{aligned} n_0'(t) &= -n_0, \\ n_k'(t) &= -n_k + n_{k-1} \qquad k = 1, \ldots, K-1, \\ n_K'(t) &= n_{K-1}. \end{aligned}$$

Then $n_0, n_1, .., n_{K-1} \to 0$ and $n_K \to 1$. Thus as $t \to +\infty$, all bidders end up bidding $\mu = 1$ (the ceiling price) as expected.

We can also check by hand that when $p \leq 1/2$ the configuration $(1 - p, p, 0, \ldots, 0)$ is a stationary state (since in that case $P_0 = p/n_0 = p/(1-p)$, $P_1 = 0$). Moreover, a standard stability analysis based on the linearization of the rate equation near this stationary state (see Appendix) shows it is locally asymptotically stable if $p > 1/2$. Similarly $(0, \ldots, 0, 1-p, p)$ is a stationary state when $p \geq 1/2$. which is locally asymptotically stable if $p > 1/2$.

In general the asymptotic behavior of $(n_0, \ldots, n_K)$ when $p \in (0,1)$ is not clear due to the large number of equations in system (3.2) and their nonlinearity through P_k. To better understand the dynamics in general it is useful to rewrite system (3.2) as a single equation that can be approximated in the continuum limit $\gamma \sim 0$ by a first-order equation easier to analyze.

3.2 Continuous limit of the rate equations as $\gamma \to 0$.

3.2.1 Continuous limit when $\sigma = 0$.

We first rewrite system (3.2) as a single equation for the empirical measure $f_t = \sum_{k=0}^{K} n_k \delta_{k\gamma}$. Here $\delta_{k\gamma}$ is the Dirac mass located at $k\gamma$. Then the action of f_t on a function ϕ, i.e., the integral of ϕ with respect to. the measure f_t, is given by $(f_t, \phi) = \int_0^1 \phi(\mu) f_t(d\mu) = \sum_{k=0}^{K} n_k \phi(k\gamma)$. We can think of ϕ as a macroscopic quantity we are interested in, being (f_t, ϕ) the mean value of this quantity at time t. For instance taking $\phi(\mu) = \mu$, $(f_t, \phi) = \sum_{k=0}^{K} n_k (k\gamma)$ is the mean value of μ.

Taking the time derivative of $(f_t, \phi) = \sum_{k=0}^{K} n_k \phi(k\gamma)$ and evaluating $n_k'(t)$ using the rate equation, we obtain through some computations (see Appendix) that

$$
\begin{aligned}
\frac{1}{\gamma}\frac{d}{dt}(f_t, \phi) =& f_t(\{0\})(1 - P_0[f_t](0))(\phi'(0) - \frac{\gamma}{2}\phi''(0)) \\
& - f_t(\{1\}) P_0[f_t](1)(\phi'(1) + \frac{\gamma}{2}\phi''(1)) \\
& + (f_t, (2P_0[f_t] - 1)\phi' + \frac{\gamma}{2}\phi'') + O(\gamma^2),
\end{aligned}
\tag{3.3}
$$

with

$$
P_0[f_t](\mu) := \begin{cases} 1_{F_t(\mu) \leq p} + \dfrac{p - f_t([0,\mu))}{f_t(\{\mu\})} 1_{\mu = F_t^{-1}(p)} & \mu \in (0,1], \\ 1_{f_t(\{0\}) \leq p} + \dfrac{p}{f_t(\{0\})} 1_{f_t(\{0\}) > p} & \mu = 0. \end{cases}
\tag{3.4}
$$

Here $F_t : [0,1] \to [0.1]$, $F_t(\mu) = f_t([0,\mu])$, is the cumulative distribution function (CdF) of f_t, and F_t^{-1} is its generalized inverse defined as $F_t^{-1}(r) = \inf\{\mu \in [0,1] : F_t(\mu) \geq r\}$, $r \in (0,1]$ (we refer to [12] for a survey of the properties of the generalized inverse).

Thus, upon rescaling time considering $t \leftarrow \gamma t$ we expect the rate equations are well approximated in the limit $\gamma \simeq 0$ by the equation given in weak form by

$$\begin{aligned} \frac{d}{dt}(f_t, \phi) = &-f_t(\{1\})P_0[f_t](1)\phi'(1) + f_t(\{0\})(1 - P_0[f_t](0))\phi'(0) \\ &+ (f_t, (2P_0[f_t] - 1)\phi') \end{aligned} \tag{3.5}$$

The first two terms account for the truncation of μ at the boundary points 0 and 1 so that μ remains always in $[0,1]$. The last term translates the learning rule $\mu \leftarrow \mu \pm \gamma$ and thus drives the dynamic in $(0,1)$.

3.2.2 Continuous limit when $\sigma > 0$.

When $\sigma > 0$ the only change with respect to the case $\sigma = 0$ lies in the winning probability of a bidder with μ. Indeed when $\sigma > 0$, a participant's bid is not exactly her μ as when $\sigma = 0$ but a random variable with distribution $N(\mu, \sigma^2)$ truncated to remain in $[0,1]$. More precisely, denoting Φ the CdF of $N(0,1)$, this truncated normal has distribution

$$f_{N(\mu,\sigma^2)|[0,1]}(db) := \Big(1 - \Phi\Big(\frac{1-\mu}{\sigma}\Big)\Big)\delta_1 + \Phi\Big(-\frac{\mu}{\sigma}\Big)\delta_0 + 1_{\{0<b<1\}} f_{N(\mu,\sigma^2)}(db)$$

where $f_{N(\mu,\sigma^2)}$ is the density of $N(\mu,\sigma^2)$. In the limit $\gamma \to 0$, the bids are then distributed as $g_{t,\sigma}(db) := \int_0^1 f_{N(\mu,\sigma^2)|[0,1]}(db)\, df_t(\mu)$. Notice this distribution is absolutely continuous in $(0,1)$ with Dirac masses at 0 and 1. Denote $G_{t,\sigma}(b)$ its CdF and $b_{t,\sigma,p} = G^{-1}_{t,\sigma}(p)$. Then a participant bidding b wins as follows:

- if $b_{t,\sigma,p} = 0 \Leftrightarrow g_{t,\sigma}(\{0\}) \geq p$, the agent then wins if $b = 0$ with probability $\frac{p}{g_{t,\sigma}(\{0\})}$,
- if $b_{t,\sigma,p} \in (0,1) \Leftrightarrow g_{t,\sigma}(\{0\}) < p \leq g_{t,\sigma}([0,1))$, the agent then wins if $b \leq b_{t,\sigma,p}$ (since $g_{t,\sigma}$ is abs.cont in $(0,1)$) with probability 1,
- if $b_{t,\sigma,p} = 1 \Leftrightarrow g_{t,\sigma}([0,1)) < p$, the agent then wins if $b < 1$ with probability 1, and also if $b = 1$ with probability $\frac{p - g_{t,\sigma}([0,1))}{g_{t,\sigma}(\{1\})}$.

Thus a bidder bidding $b \sim N(\mu, \sigma^2)$ truncated to remain in $[0,1]$ wins with probability

$$\begin{aligned} P_\sigma[f_t](\mu) := &1_{\{b_{t,\sigma,p}=0\}} \frac{p}{g_{t,\sigma}(\{0\})} \Phi\Big(-\frac{\mu}{\sigma}\Big) + 1_{\{0<b_{t,\sigma,p}<1\}} \Phi\Big(\frac{b_{t,\sigma,p} - \mu}{\sigma}\Big) \\ &+ 1_{\{b_{t,\sigma,p}=1\}} \Big\{ \Phi\Big(\frac{1-\mu}{\sigma}\Big) + \Big(1 - \Phi\Big(\frac{1-\mu}{\sigma}\Big)\Big) \frac{p - g_{t,\sigma}([0,1))}{g_{t,\sigma}(\{1\})} \Big\}. \end{aligned}$$

We then obtain as before the following equation for f_t:

$$\begin{aligned} \frac{d}{dt}(f_t, \phi) = &-f_t(\{1\})P_\sigma[f_t](1)\phi'(1) + f_t(\{0\})(1 - P_\sigma[f_t](0))\phi'(0) \\ &+ (f_t, (2P_\sigma[f_t] - 1)\phi'). \end{aligned} \tag{3.6}$$

Notice that in the limit $\sigma \sim 0$, we recover (3.5) using that $\Phi\left(\frac{b-\mu}{\sigma}\right) \sim 1_{\mu<b}$, $b_{p,\sigma} \sim p$. The main difference between the cases $\sigma = 0$ and $\sigma > 0$ lies in the fact that when $\sigma > 0$ the bids are distributed according to an absolutely continuous distribution in $(0, 1)$ whereas, when $\sigma = 0$, the bids are the μ's and thus are distributed according to f_t which may have Dirac masses in $(0, 1)$. The additional term $1_{\mu=F_t^{-1}(p)}$ in $P_0[f_t]$ accounts for the possibility of a Dirac mass at $F_t^{-1}(p)$. We will see below that this additional term is crucial to understand the long-time behaviour of f_t and is ultimately responsible for the phase transition occurring at $p = \frac{1}{2}$.

3.3 A kinetic approach

We can also analyze the dynamic adopting a kinetic approach as presented e.g. in [26]. This framework comes from the analysis of rarefied gases where the distribution of position and velocity of the particles conforming the gas obey the famous Boltzmann equation. This methodology has been successfuly adapted over the last 20 years to model socio-economic problems (see [26]) and in evolutionary game theory in [31].

In our setting, the distribution f_t of parameter μ satisfies the following integrodifferential equation

$$\frac{d}{dt}\int_0^1 \phi(\mu) f_t(d\mu) = \int_0^1 \mathbb{E}[\phi(\mu') - \phi(\mu)]\, f_t(d\mu) \tag{3.7}$$

for any $\phi \in C([0, 1[)$, where μ' denotes the value of μ parameter at the end of the round after the bidder up-dated its μ according to (2.1). Recalling that $P_\sigma[f_t](\mu)$ is the probability that a bidder with μ wins at time t, we have

$$\begin{aligned}\mathbb{E}[\phi(\mu') - \phi(\mu)] = {} & [\phi(\min\{\mu + \gamma, 1\}) - \phi(\mu)]P_\sigma[f_t](\mu) \\ & + [\phi(\max\{\mu - \gamma, 0\}) - \phi(\mu)](1 - P_\sigma[f_t](\mu)).\end{aligned} \tag{3.8}$$

Thus equation (3.7) characterizes the evolution of f_t through the evolution of $\int_0^1 \phi(\mu) f_t(d\mu)$, the expected value of ϕ at time t, averaging over all the possible jumps of ϕ due to the learning process.

An argument based on the classical Banach fixed-point theorem (see e.g., [9] or [29]) gives

Theorem 3.1. *For any initial condition $f_0 \in P([0, 1])$ there exists a unique solution $f \in C([0, +\infty), P([0, 1])) \cap C^1((0, +\infty), P([0, 1]))$ to equation (3.7).*

Here $P([0, 1])$ is endowed with the total variation norm defined as

$$\|f\|_{TV} = \sup_\phi \int_0^1 \phi(\mu)\, f(d\mu)$$

where the sup is taken over all $\phi \in C([0, 1])$, $\|\phi\|_\infty \leq 1$.

Unfortunately studying the long-time behaviour of a solution to an integro-differential equations like (3.7) is in general not easy. A well-known procedure in statistical mechanics called grazing limit permits approximating it by a differential equation. It consists of taking a new time scale, or equivalently, γ small, and writing

$$\begin{aligned}\mathbb{E}[\phi(\mu') - \phi(\mu)] \approx \phi'(\mu)\Big\{&(\min\{\mu+\gamma,1\} - \mu)P_\sigma[f_t](\mu) \\ &+ (\max\{\mu-\gamma,0\} - \mu)(1 - P_\sigma[f_t](\mu)\Big\}.\end{aligned} \tag{3.9}$$

To focus on the dynamic in $(0,1)$, we suppose that ϕ is supported in $(0,1)$. Then for γ small, this expression can be further simplified:

$$\begin{aligned}\mathbb{E}[\phi(\mu') - \phi(\mu)] &\approx \phi'(\mu)\Big\{((\mu+\gamma) - \mu)P_\sigma[f_t](\mu) \\ &\qquad + ((\mu-\gamma) - \mu)(1 - P_\sigma[f_t](\mu)\Big\} \\ &= \gamma\phi'(\mu)(2P_\sigma[f_t](\mu) - 1).\end{aligned} \tag{3.10}$$

Returning to (3.7) we obtain the equation

$$\frac{1}{\gamma}\frac{d}{dt}\int_0^1 \phi(\mu)f_t(d\mu) = \int_0^1 \phi'(\mu)(2P_\sigma[f_t](\mu) - 1)f_t(d\mu) \tag{3.11}$$

for any $\phi \in C^1([0,1[)$, Changing the time scale to $t \leftarrow \gamma t$, we thus recovered (3.6) in the limit $\gamma \to 0$.

This informal derivation can be justified (see proof in Appendix):

Theorem 3.2. *Given an initial condition $f_0 \in P([0,1])$, denote f^γ the unique solution to* (3.7) *given by Theorem 3.1. We change the time scale to $\tau = \gamma t$ considering (with a slight abuse of notation) $f^\gamma_\tau := f^\gamma_t$. Then there exists $f \in C([0,+\infty), P([0,1]))$ such that, along a subsequence $\gamma \to 0$, f^γ converges to f in $C([0,T], P([0,1]))$ for any $T > 0$.*

Here $P([0,1])$ is endowed with the weak convergence topology i.e., the measures $f_n \in P([0,1])$ converge to $f \in P([0,1])$ if $\int_0^1 \phi(\mu)f_n(d\mu) \to \int_0^1 \phi(\mu)f(d\mu)$ for any $\phi \in C([0,1])$. Notice this convergence is weaker than the TV-norm used in Theorem 3.1.

In view of (3.11) f^γ_τ satisfies the approximate equation

$$\frac{d}{d\tau}\int_0^1 \phi(\mu)f^\gamma_\tau(d\mu) \approx \int_0^1 \phi'(\mu)(2P_\sigma[f^\gamma_\tau](\mu) - 1)f^\gamma_\tau(d\mu)$$

for ϕ C^1 supported in $(0,1)$. It is tempting to pass to the limit $\gamma \to 0$ to deduce that the limit of the f^γ given by Theorem 3.2. It is however not obvious to pass to the limit $\gamma \to 0$ in the right-hand side due to the low regularity of $P_\sigma[f^\gamma_\tau]$ with respect to f^γ_τ.

3.4 Analysis of the continuous equation.

In the previous sections we deduced informally the following continuous equation for the distribution of μ's in the limit $\gamma \to 0$:

$$\begin{aligned}\frac{d}{dt}(f_t, \phi) = f_t(\{0\})(1 - P_\sigma[f_t](0))\phi'(0) - f_t(\{1\})P_\sigma[f_t](1)\phi'(1) \\ + (f_t, (2P_\sigma[f_t] - 1)\phi').\end{aligned} \tag{3.12}$$

We obtained this equation as a limit of the rate equations system (3.2) and also as an approximation of the integro-differential equation (3.7).

We study in this section the long-time behavior of the solution f_t to equation (3.12).

Let us first examine the intuitive content of each of the three terms in the right-hand side. The first two terms with $\phi'(0)$ and $\phi'(1)$ comes from the fact that we imposed in the updating rule that μ must remain higher than or equal to the cost (here 0) and lower than or equal to the ceiling price (here 1). Mathematically they force the solution f_t to remain supported in $[0, 1]$. The last term $(f_t, (2P_\sigma[f_t] - 1)\phi')$ is the translation of the updating rule $\mu' = \mu \pm \gamma$, i.e., of the learning process, and thus is the most interesting term.

When $p = 1$ (respectively $p = 0$) we already saw using the rate equations (3.2) that all bidders end up with $\mu = 1$ (respectively $\mu = 0$). We can recover this result from (3.12). Indeed $P_\sigma \equiv 1$ for $\sigma \geq 0$ (for $\sigma > 0$ notice that $b_{t,\sigma,p} = 1$ necessarily). Thus (3.12) becomes

$$\frac{d}{dt}(f_t, \phi) = -f_t(\{1\})\phi'(1) + (f_t, \phi').$$

Notice that when $f_t = \delta_1$, the right-hand side equals 0 so that a Dirac's mass δ_1 is a stationary solution of this equation. To see that any solution f_t converges to δ_1 it is enough to verify that its mean $m(t) = (f_t, \mu)$ converges to 1. Taking $\phi(\mu) = \mu$ we obtain $m'(t) = 1 - f_t(\{1\})$. Thus $m'(t) > 0$, i.e., m is strictly increasing, unless $f_t(\{1\}) = 1$, which holds if and only if $f_t = \delta_1$, that is, if and only if $m = 1$. We thus obtain $f_t \to \delta_1$ as expected.

From now on we suppose that $p \in (0, 1)$.

We begin our general study of (3.12) noticing that equation (3.12) is invariant under the change $(p, \mu) \to (1-p, 1-\mu)$. Indeed denoting f_t the solution for p and g_t the solution for $1 - p$, g_t is the symmetric of f_t with respect to $\mu = \frac{1}{2}$ (see Appendix for the proof):

Proposition 2. *Let f_t be a solution of (3.12) for a proportion p of winners. Define the measure g_t by*

$$\int \phi(1 - \mu)g_t(d\mu) = \int \phi(\mu)f_t(d\mu) \qquad \text{for any } \phi.$$

Then g_t is solution of (3.12) for a proportion $1 - p$ of winners.

From now on we assume for simplicity that $\sigma = 0$ so that we study equation

$$\frac{d}{dt}(f_t, \phi) = f_t(\{0\})(1 - P_0[f_t](0))\phi'(0) - f_t(\{1\})P_0[f_t](1)\phi'(1) + (f_t, (2P_0[f_t] - 1)\phi'). \tag{3.13}$$

Notice that general well-posedness results for equations given in weak form by

$$\frac{d}{dt}(f_t, \phi) = (f_t, v[f_t]\phi')$$

for any ϕ, i.e.,

$$\partial_t f_t + \partial_\mu(v[f_t]f_t) = 0,$$

usually requires the vector field $(t, \mu) \to v[f_t](\mu)$ to be Lipschitz plus some regularity of v with respect to f_t (see e.g. [3]). Here however we cannot directly apply existing results to equation (3.13) due to two difficulties. The first issues are the boundary condition at 0 and 1, namely

$$f_t(\{0\})(1 - P_0[f_t](0))\phi'(0) - f_t(\{1\})P_0[f_t](1)\phi'(1)$$

which have not been considered so far in the partial differential equation literature, even though the dynamics were introduced by Feller in the 1950s. The second difficulty is the low regularity of the vector field $2P_0[f_t] - 1$. Indeed, rewriting the definition (3.4) of $P_0[f_t](\mu)$, $\mu \in (0, 1]$, as

$$P_0[f_t](\mu) = \begin{cases} 0 & \text{if } \mu > F_t^{-1}(p) \\ 1 & \text{if } \mu < F_t^{-1}(p) \text{ or } \mu = F_t^{-1}(p),\ f_t(\{\mu\}) = 0 \\ \frac{p - f_t([0,\mu])}{f_t(\{\mu\})} & \text{if } \mu = F_t^{-1}(p),\ f_t(\{\mu\}) > 0 \end{cases}$$

we can see that $P_0[f_t]$ is not even continuous. Moreover the regularity of $P_0[f_t]$ with respect to f_t, required to complete the justification of the grazing limit procedure presented before, is a delicate issue. On the other hand, two facts are in favor of well-posedness. First equation (3.13) appeared naturally as limit of the rate equations system (3.2) and the integro-differential equation (3.7). Moreover $P_0[f_t](\mu)$ is discontinuous only at $\mu = F_t^{-1}(p)$ but is very smooth in $\{\mu < F_t^{-1}(p)\}$ and in $\{\mu > F_t^{-1}(p)\}$. In fact the measures f_t^+ and f_t^- obtained restricting f_t to $[0, F_t^{-1}(p))$ and $(F_t^{-1}(p), 1]$ respectively satisfy

$$\frac{d}{dt}(f_t^+, \phi) = f_t^+(\{0\})(1 - P_0[f_t](0))\phi'(0) + (f_t^+, \phi')$$

and

$$\frac{d}{dt}(f_t^-, \phi) = -f_t^-(\{1\})P_0[f_t](1)\phi'(1) - (f_t^-, \phi')$$

as can be seen from (3.13) taking ϕ supported in $[0, F_t^{-1}(p))$ and $(F_t^{-1}(p), 1]$. In particular as long as $F_t^{-1}(p) \in (0, 1)$, $f_t^\pm$ are then solutions of the following simple transport equation

$$\partial_t f_t^\pm \pm \partial_\mu f_t^\pm = 0$$

on the time-varying interval $[0, F_t^{-1}(p))$ and $(F_t^{-1}(p), 1]$. Notice that f_t^+ and f_t^- are transported to the right and to the left respectively at speed 1. When mass coming from these two measures meet, a Dirac mass builds up at $F_t^{-1}(p)$ and moves, but it is not clear at which rate.

So we postpone the non-trivial issue of well-posedness of equation (3.13) to a future work and will focus instead on the qualitative behavior of the solutions assuming equation (3.13) is well-posed. The results presented in the remaining part of the paper completely agree with the numerical simulations.

We begin by providing the exact and explicit expression of the solutions to (3.12) for some particular initial condition. We then study the dynamic of the mean value of μ and conclude with the qualitative study of the evolution of any solution.

3.4.1 Exact solution for some particular cases

We can explicitly solve equation (3.13) in some particular cases. We first consider the case where f_0 is a Dirac mass located at a point in $(0, 1)$. We then consider the case where f_0 is the uniform distribution over $[0, 1]$. In both cases we will see that the solution goes toward δ_0 or δ_1 depending on whether $p < 1/2$ or $p > 1/2$. We will give the solution up to the first time T where $f_T = \delta_0$ or $f_T = \delta_1$. We conclude with some remarks concerning the solution starting from δ_0 or δ_1.

First when f_0 is a Dirac mass at a point in $(0, 1)$ (i.e., all bidders use the same μ initially) then (see Appendix)

Proposition 3. *If $f_0 = \delta_{\mu(0)}$ with $\mu(0) \in (0, 1)$ then the solution of (3.13) is given by*

$$f_t = \delta_{\mu(t)}, \qquad \mu(t) = (2p - 1)t + \mu(0),$$

as long as $\mu(t) \in (0, 1)$ i.e., for $t \in [0, T]$ with $T = +\infty$ if $p = \frac{1}{2}$, $T = \frac{\mu(0)}{1-2p}$ if $p < \frac{1}{2}$, $T = \frac{1-\mu(0)}{2p-1}$.

Notice that $\mu(t)$ increases to 1 or decreases to 0 according to whether $p > 1/2$ or $p < 1/2$, whereas it is stationary when $p = 1/2$.

When f_0 is the uniform distribution over $[0, 1]$, which is the case shown in Figures 1. From these simulations we can conjecture the evolution of f_t can be decomposed into three phases. First f_t becomes narrower at a constant rate while a Dirac mass at $\mu = p$ builds up. This goes on until the mass completely disappears on one side of δ_p. Then the remaining mass on the other side keeps accumulating on the Dirac mass until f_t is only a Dirac mass. From that moment on, the evolution is the same as that given in the previous proposition. The next result shows this intuition is correct (remember we give the solution up to the first time T where $f_T = \delta_0$ or $f_T = \delta_1$).

Proposition 4. *When $f_0 = 1$ in $[0, 1]$, the solution of (3.13) is given by*

- *if* $p \le \frac{1}{4}$*:*

$$f_t = \begin{cases} 1_{[t,1-t]} + 2t\delta_{a(t)}, & \\ \qquad a(t) = p & 0 \le t \le p \\ 1_{[a(t),1-t]} + 2\sqrt{pt}\delta_{a(t)}, & \\ \qquad a(t) = 2\sqrt{pt} - t, & p \le t \le T := 4p. \end{cases}$$

- *if* $\frac{1}{4} < p < \frac{1}{2}$*:*

$$f_t = \begin{cases} 1_{[t,1-t]} + 2t\delta_{a(t)}, & \\ \qquad a(t) = p & 0 \le t \le p \\ 1_{[a(t),1-t]} + (t + a(t))\delta_{a(t)}, & \\ \qquad a(t) = 2\sqrt{pt} - t, & p \le t \le \frac{1}{4p}, \\ \delta_{a(t)}, & \\ \qquad a(t) = (2p-1)(t - \frac{1}{4p}) + 1 - \frac{1}{4p}, & \frac{1}{4p} \le t \le T := \frac{1}{2(1-2p)}. \end{cases}$$

- *if* $p = \frac{1}{2}$*:*

$$f_t = \begin{cases} 1_{[t,1-t]} + 2t\delta_{a(t)}, & \\ \qquad a(t) = p & 0 \le t \le \frac{1}{2} \\ \delta_{a(t)}, & \\ \qquad a(t) = \frac{1}{2} & t \ge \frac{1}{2} \end{cases}$$

- *if* $\frac{1}{2} \le p \le \frac{3}{4}$*:*

$$f_t = \begin{cases} 1_{[t,1-t]} + 2t\delta_{a(t)}, & \\ \qquad a(t) = 1-p & 0 \le t \le 1-p \\ 1_{[t,a(t)]} + 2\sqrt{(1-p)t}\delta_{a(t)}, & \\ \qquad a(t) = 1 - 2\sqrt{(1-p)t} + t, & 1-p \le t \le \frac{1}{4(1-p)}, \\ \delta_{a(t)}, & \\ \qquad a(t) = -(1-2p)(t - \frac{1}{4(1-p)}) + \frac{1}{4(1-p)}, & \frac{1}{4(1-p)} \le t \le T := \frac{1}{2(2p-1)}. \end{cases}$$

- *if* $\frac{3}{4} \le p \le 1$*:*

$$f_t = \begin{cases} 1_{[t,1-t]} + 2t\delta_{a(t)}, & \\ \qquad a(t) = p & 0 \le t \le 1-p \\ 1_{[t,a(t)]} + 2\sqrt{(1-p)t}\delta_{a(t)}, & \\ \qquad a(t) = 1 - 2\sqrt{(1-p)t} + t, & 1-p \le t \le T := 4(1-p). \end{cases}$$

The proof is given in the Appendix.

We show in figure 1 (top row) the very good agreement between agent based simulation (blue lines) with $p = 1/4$ (left) and $p = 0.8$ (right) with μ initially uniformly distributed in $[0,1]$, and the theoretical formula above (the red points indicates the location $a(t)$ of the Dirac mass).

Eventually let us examine what happens near $\mu = 0$ and $\mu = 1$. By the symmetry of the problem, we only need to focus on $\mu = 0$. We already saw with the rate equations that $(1-p, p, 0, \dots, 0)$ is a locally asymptotically stable stationary state when $p < 1/2$. This is what matters with respect to. numerical simulations. Concerning the continuous equation, the situation is less clear. Notice that δ_0 is not a stationary state since the right-hand side of (3.13) is $P_0[\delta_0](0)\phi'(0) = p\phi'(0) \neq 0$. Based on the rate equations, we conjecture there is a unique stationary state supported near 0. Though we could not find it explicitly, notice that $f = (1-p)\delta_0 + p\delta_\gamma$ is an approximate stationary state of the continuous equation (3.13) up to an error of order γ. Indeed noticing $P_0[f](0) = p/(1-p)$ and $P_0[f](\mu) = 0$, $\mu > 0$, the right hand side of (3.13) is

$$f(\{0\})P_0[f](0))\phi'(0) - p(\delta_\gamma, \phi') = p(\phi'(0) - \phi'(\gamma)) = O(\gamma).$$

We can mimick the linear stability analysis done for the rate equation (though less rigorously since there is no available theory of stability for equation like (3.13)) to show that f is locally asymptotically stable if $p < 1/2$ (see Appendix). Thus when $f_0 = \delta_0$ the solution f_t goes to this stationary state when $p < 1/2$.

3.4.2 Asymptotic behavior for any initial condition.

The particular cases studied in the previous sections explains the phase transition at $p = 1/2$ observed in the numerical simulations when f_0 is either a Dirac mass in $(0,1)$ or the uniform distribution over $(0,1)$. Indeed in both cases, f_t converges to the stationary state located near δ_0 if $p < 1/2$ or δ_1 if $p > 1/2$ (the stationary state being $(1-p, p, 0, \dots, 0)$ and $(0, \dots, 0, 1-p, p)$). In this section we will see that this true for any initial condition a give a qualitative description of the evolution of any solution f_t.

Recall that given a solution f_t we denote $F_t(\mu) = f_t([0,\mu])$, $\mu \in [0,1]$, its cumulative distribution function (i.e., $F_t(\mu)$ is the fraction of bidders bidding less thn or equal to μ at time t). The generalized inverse of F_t is then $F_t^{-1}(r) = \inf\{\mu \in [0,1] : F_t(\mu) \geq r\}$, $r \in (0,1]$.

Given a solution f_t of (3.12) with $\sigma = 0$, it is useful to study its moment. We could find a closed equation only for the first moment $m(t) = \int \mu \, f_t(d\mu)$ i.e., the mean value of μ at time t. Indeed taking $\phi(\mu) = \mu$ in (3.12), we obtain the following equation for m:

$$m'(t) = -f_t(\{1\})P_0[f_t](1) + f_t(\{0\})(1 - P_0[f_t](0)) + (f_t, (2P_0[f_t] - 1)).$$

We can simplify the right-hand side to obtain (see Appendix):

$$m'(t) = \begin{cases} 2p - 1 & \text{if } F_t^{-1}(p) \in (0,1), \\ p - f_t(\{1\}) & \text{if } F_t^{-1}(p) = 1, \\ f_t(\{0\}) - (1-p) & \text{if } F_t^{-1}(p) = 0. \end{cases} \tag{3.14}$$

Thus while $F_t^{-1}(p)$ remains in $(0,1)$, m increases or decreases according to whether $p > 1/2$ or $p < 1/2$. This is other evidence for a phase transition at $p = 1/2$.

Next notice that studying the evolution of f_t is equivalent to studying the evolution of $F_t^{-1}(r)$ for $r \in (0,1]$. For instance $F_t^{-1}(1)$ is the right end of *supp* f_t and $F_t^{-1}(0^+) := \lim_{r\downarrow 0} F_t^{-1}(r)$ is its left end. It turns out that studying $F_t^{-1}(r)$ can be simpler. Indeed since the vector field $2P_0[f_t](\mu) - 1$ 'moves' f_t in $(0,1)$, according to (3.13), we have

$$\partial_t F_t^{-1}(r) = 2P_0[f_t](F_t^{-1}(r)) - 1$$

while $F_t^{-1}(r) \in (0,1)$ (see, e.g., [29]). Notice in particular that if $r < p < s$ with $F_t^{-1}(r) < F_t^{-1}(p) < F_t^{-1}(s)$ then $P_0[f_t]((F_t^{-1}(r)) = 1$ and $P_0[f_t](F_t^{-1}(s)) = -1$ so that

$$\partial_t F_t^{-1}(r) = 1 \qquad \text{and} \qquad \partial_t F_t^{-1}(s) = -1. \tag{3.15}$$

This is the mathematical translation of the fact that bidders bidding less than p (respectively more than p) increase (respectively, decrease) their bid by γ, i.e., by 1 on the time scale γt (the time scale used in equation (3.13)). This is clear on the simulations shown in Figure 1 (top row) where straight lines of slope ± 1 (on the time scale γt) can be clearly seen.

We can thus obtain the following qualitative behaviour of a solution f_t to equation (3.13) starting from an initial condition f_0. We assume that $F_0^{-1}(p) \in (0,1)$ so that $F_0^{-1}(0^+) < F_0^{-1}(p) < F_0^{-1}(1)$. Then by (3.15), the support of f_t shrinks around $F_t^{-1}(p)$ up to some time T where either f_T is a Dirac mass, in which case the evolution for $t \geq T$ is given by Prop. 3, or f_T is supported in $[F_T^{-1}(p), F_T^{-1}(1)]$ or $[F_T^{-1}(0^+), F_T^{-1}(p)]$. Assume for instance that *supp* $f_T \subset [F_T^{-1}(p), F_T^{-1}(1)]$ with $F_T^{-1}(1) > F_T^{-1}(p) > 0$, i.e., there is a Dirac mass at $F_T^{-1}(p)$ with $1 > f_T(\{F_T^{-1}(p)\}) \geq p$ (all the mass initially on the left of $F_0^{-1}(p)$) collapsed to a Dirac mass). For $t \geq T$, since $\partial_t F_t^{-1}(1) = -1$, $F_t^{-1}(1)$ decreases. If $p \geq 1/2$ then $P_0[f_t](F_t^{-1}(p)) = p/f_t(\{F_t^{-1}(p)\}) \geq 1/2$. Thus, if $p \geq 1/2$, at some time T', f_t becomes a Dirac mass necessarily located at $m(0)$ (since $m(t)$ is constant). If $p < 1/2$, $F_t^{-1}(1)$ decreases until a time T' where either $f_{T'}$ is a Dirac mass in $[0,1)$ (whose posterior evolution is given by Prop. 3), or *supp* $f_{T'} = [0, F_{T'}^{-1}(1)]$ with $F_{T'}^{-1}(1) > 0$ i.e., $f_{T'}(\{0\}) \in (p,1)$. In the latter case, $F_t^{-1}(1)$ keeps on decreasing until f_t becomes a Dirac mas located at a point near 0. Since $p < 1/2$, f_t then converges to the unique stationary state near 0 (approximately given by $(1-p)\delta_0 + p\delta_\gamma$).

Summing up this analysis we proved the following the result:

Theorem 3.3. *Given an initial distribution f_0 of μ such that $F_0^{-1}(p) \in (0,1)$ (i.e., $f_0(\{0\}) < p$ and $f_0(\{1\}) < 1-p$), the solution f_t of (3.13) converges to the Dirac mass $\delta_{m(0)}$ at the initial mean μ value when $p = 1/2$, and to the unique stationary state near 0 (respectively, near 1) when $p < 1/2$ (respectively, $p > 1/2$) approximately given by $(1-p)\delta_0 + p\delta_\gamma$ (respectively, $p\delta_1 + (1-p)\delta_{1-\gamma}$).*

4 Conclusion

We propose in this paper a partial differential equations analysis of learning in auctions. We apply it a simple model of a procurement auction taking place in several rounds where a large number of bidders offer at each round a fixed volume v of energy. In each round only a given proportion $1/\rho$ of participants win. We can then think of ρ as the competition level of the round. The participants' bidding behavior, i.e., the price they offer for a volume v, is characterized by two parameters, namely the mean bid value μ and its deviation. From round to round, participants myopically adjust their mean bid value μ: they raise it (respectively, lower it), by a constant increment γ if they are among the winners (respectively, losers) of the round.

This model was proposed in [32] where numerical simulations showed the presence of a phase transition at $\rho = 2$: prices increase to the upper limit when $\rho > 2$, whereas they drop to the inferior limit when $\rho < 2$. In this paper we proposed a theoretical analysis based on rate equations and their continuous limit when $\gamma \sim 0$. The resulting equation is a first order, non-linear, partial differential equation for the distribution $f_t(d\mu)$ in the limit $\gamma \to 0$ on the time scale γt. This equation presents unusual boundary conditions and is quite challenging to study due to the low regularity of the vector field driving the dynamic in $(0, 1)$. Nevertheless it can be solved explicitly when f_0 is a Dirac mass and when it is uniform on $[0, 1]$. In general, rewriting it using the generalized inverse of the cumulative distribution function of f_t permits obtaining the qualitative behaviour of f_t as $t \to +\infty$. This study confirms and explains the phase transition numerically observed in [32].

According to this result, whatever the initial distribution of μ in the population, the myopic learning procedure leads to an asymptotic perfect coordination of the bidders in the sense that as time passes, they tend to share the same value of μ. The limit value is either closed to the cost of the participants (here 0) or the ceiling price (here 1) depending on the competition level ($\rho > 2$ in the first case, $\rho < 2$ in the second case). More precisely if there are too many winners, more than half the number of participants, then the μ values decrease to near the cost. On the other hand, if there are too few winners, less than half the number of participants, then the μ values increase to near the ceiling price.

This result has concrete consequences. Indeed, it is well known in the literature concerning RE auctions that a high level of participation is crucial to obtaining good energy prices. Our model shows that is indeed the level of competition (i.e., the offered volume in relation to the auctioned volume) that really matters. Moreover the phase transition identified at $\rho = 2$ precisely quantifies the critical role played by the level of competition. Since in this study the bidders bid approximately their μ, to guarantee good prices (i.e., low prices), the auctioneer must set the competition level $\rho > 2$. Notice eventually that the phase transition at $\rho = 2$ has been observed in real RE auctions (see the discussion in [32].

As a final comment, notice that the use of *partial* differential equations is uncommon in the mathematical literature on auctions theory. We believe however that it can bring valuable insights in an evolutionary model of auctions involving a large number of agents able to learn from round to round.

Declarations

Credit authorship contribution statement: all the authors contributed on the conceptualization and analysis of the problem; and writing, reviewing and editing of the manuscript; NS and JPP contributed on the design of the computational simulations, and the derivation of the theoretical results.

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Appendix

4.1 Well-posedness of the rate equations: proof of Prop. 1.

We first verify that the right-hand side of system (3.2) is globally Lipschitz in $(n_0, \dots, n_K)$. Indeed recalling the definition of P_k, we have for $k = 1, \dots, K-1$ that

$$\begin{aligned} & n_{k-1}P_{k-1} + n_{k+1}(1 - P_{k+1}) - n_k \\ &= \begin{cases} n_{k-1} - n_k & \text{if } F_{k+1} \le p, \\ n_{k-1} - n_k + n_{k+1} - p + F_k & \text{if } F_k \le p < F_{k+1}, \\ n_{k-1} - n_k + n_{k+1} & \text{if } F_{k-1} \le p < F_k, \\ p - F_{k-2} - n_k + n_{k+1} & \text{if } F_{k-2} \le p < F_{k-1}, \\ -n_k + n_{k+1} & \text{if } p < F_{k-2}, \end{cases} \end{aligned} \tag{4.1}$$

Notice the right-hand side is Lipschitz being continuous in all $\mathbb{R}^{K+1}$ and linear in regions of $\mathbb{R}^{K+1}$ delimited by hyperplanes. The same reasoning applies to

$$-n_0P_0 + n_1(1 - P_1) = \begin{cases} -n_0 & \text{if } n_0 + n_1 \le p, \\ n_1 - p & \text{if } n_0 + n_1 > p, \end{cases} \tag{4.2}$$

and

$$n_{K-1}P_{K-1} - n_K(1 - P_K). = \begin{cases} n_{K-1} & \text{if } F_K \le p, \\ n_{K-1} - n_K + p - F_{K-1} & \text{if } F_{K-1} \le p < F_K, \\ p - F_{K-2} - n_K & \text{if } F_{K-2} \le p < F_{K-1}, \\ -n_K & \text{if } p < F_{K-2}. \end{cases} \tag{4.3}$$

We conclude that the right-hand side of system (3.2) is globally Lipschitz. According to the Cauchy-Lipschitz Theorem, system (3.2) has thus a unique solution for any initial condition which is defined for all time $t \in \mathbb{R}$.

Consider now an initial condition such that $(n_0 + \dots + n_K)_{|t=0} = 1$ and $n_k(0) \ge 0$, $k = 0, \dots, K$. Summing all the equations in (3.2), we see that $(n_0 + \dots + n_K)' = 0$ so that $n_0(t) + \dots + n_K(t) = 1$ for any $t \in \mathbb{R}$.

Eventually to prove that $n_k(t) \ge 0$ for any $k = 0, \dots, K$ and any $t \in \mathbb{R}$, it suffices to prove that the vector field defining the right-hand side of system (3.2) points inside the orthant $\{n \in \mathbb{R}^{K+1} : n_0, \dots, n_K \ge 0\}$ at any point of its boundary, i.e., its k-th component is ≥ 0 when $n_k = 0$ with $n_l \ge 0$, $l \ne 0$. This is clear in view of the expressions set forth above (4.1)-(4.3).

4.2 Linear stability analysis of the rate equations.

Let us show the stationary state $(1 - p, p, 0, \dots, 0)$ is locally asymptotically stable for $p < 1/2$. Notice that if $p < 1/2$ i.e., $1 - p > p$, then in a neighborhood of $(1-p, p, 0, \ldots, 0)$ we have $P_0 = p/n_0$ and $P_k = 0$, $k \geq 1$. Thus, in that neighborhood, the rate equations are

$$\begin{aligned} n_0'(t) &= -p + n_1, \\ n_1'(t) &= p - n_1 + n_2, \\ n_k'(t) &= -n_k + n_{k+1} \qquad k = 2, \ldots, K-1, \\ n_K'(t) &= -n_K. \end{aligned}$$

Notice $\tilde{n} := (n_1, \ldots, n_K)$ solves $\tilde{n}' = \tilde{A} + \tilde{B}$ where $\tilde{B} = (p, 0, \ldots, 0)^T$ and $\tilde{A} = (\tilde{a}_{ij})_{i,j=1,\ldots,K}$ is the matrix whose coefficients are all zero except $\tilde{a}_{ii} = -1$, $i \geq 1$, and $a_{i,i+1} = 1$, $i = 1, \ldots, K-1$. Thus $n_1(t) \to p$ and $n_2, \ldots, n_K \to 0$ as $t \to +\infty$ exponentially fast. Since $n_1 + n_1 + \ldots + n_K = 1$ we deduce $n_0(t) \to 1-p$.

4.3 Deduction of the continuous equation (3.3).

To obtain equation (3.3), we take the time derivative of $(f_t, \phi) = \sum_{k=0}^{K} n_k(t)\phi(k\gamma)$ and use the rate equation to evaluate $n_k'(t)$. We obtain

$$\begin{aligned} \frac{d}{dt}(f_t, \phi) &= \sum_{k=0}^{K} n_k'(t)\phi(k\gamma) \\ &= (-n_0 P_0 + n_1(1-P_1))\phi(0) + (n_{K-1}P_{K-1} - n_K(1-P_K))\phi(1) \\ &\quad + \sum_{k=1}^{K-1} (n_{k-1}P_{k-1} + n_{k+1}(1-P_{k+1}) - n_k)\phi(k\gamma) \\ &= (n_0(1-P_0) + n_1(1-P_1))\phi(0) + (n_{K-1}P_{K-1} + n_K P_K)\phi(1) \\ &\quad + \sum_{k=0}^{K-2} n_k P_k \phi((k+1)\gamma) + \sum_{k=2}^{K} n_k(1-P_k)\phi((k-1)\gamma) - (f_t, \phi). \end{aligned}$$

i.e.

$$\begin{aligned} \frac{d}{dt}(f_t, \phi) = n_0(1-P_0)\phi(0) + n_K P_K \phi(1) - n_K P_K \phi(1+\gamma) - n_0(1-P_0)\phi(-\gamma) \\ + \sum_{k=0}^{K} n_k P_k \phi(k\gamma + \gamma) + \sum_{k=0}^{K} n_k(1-P_k)\phi(k\gamma - \gamma) - (f_t, \phi). \end{aligned} \tag{4.4}$$

Notice that we can recover the equation for $n_k(t)$ simply taking ϕ to be zero outside $(k\gamma - \gamma/2, k\gamma + \gamma/2)$ and $\phi(k\gamma) = 1$. Thus the whole system of rate equations is equivalent to (4.4) for any ϕ.

Next to approximate this equation in the limit $\gamma \sim 0$, we write

$$\phi((k \pm 1)\gamma) = \phi(k\gamma) \pm \phi'(k\gamma)\gamma + \frac{\gamma^2}{2}\phi''(k\gamma) + O(\gamma^3).$$

Replacing in (4.4) we obtain

$$\begin{aligned}\frac{1}{\gamma}\frac{d}{dt}(f_t,\phi) &= -n_K P_K \phi'(1) - \frac{\gamma}{2} n_K P_K \phi''(1)\\ &\quad + n_0(1-P_0)\phi'(0) - \frac{\gamma}{2} n_0(1-P_0)\phi''(0)\\ &\quad + \sum_{k=0}^{K}(2P_k-1)n_k\phi'(k\gamma) + \frac{\gamma}{2}(f_t,\phi'') + O(\gamma^2).\end{aligned}$$

We deduce (3.3) rewriting the right hand side noticing that $F_k = f_t([0,k\gamma])$ so that $P_k = P_0[f_t](k\gamma)$ where

$$P_0[f_t](\mu) := \begin{cases} 1_{F_t(\mu)\le p} + \frac{p - f_t([0,\mu))}{f_t(\{\mu\})} 1_{\mu = F_t^{-1}(p)} & \mu \in (0,1],\\ 1_{f_t(\{0\})\le p} + \frac{p}{f_t(\{0\})} 1_{f_t(\{0\})>p} & \mu = 0.\end{cases}$$

Here F_t is the cumulative distribution function of f_t and F_t^{-1} is its generalized inverse defined as $F_t^{-1}(r) = \inf\{\mu \in [0,1] : F_t(\mu) \ge r\}$, $r \in (0,1]$.

4.4 Grazing limit: proof of Thm. 3.2.

We follow the lines of [37] and [29]. From (3.7) we can write for any $t > 0$ and any function $\phi : [0,1] \to \mathbb{R}$ 1-Lipschitz that

$$\begin{aligned}\frac{d}{dt}\int_0^1 \phi(\mu) f_t^\gamma(d\mu) &= \int_0^1 \mathbb{E}[\phi(\mu') - \phi(\mu)] f_t^\gamma(d\mu)\\ &\le \int_0^1 \mathbb{E}[|\mu'-\mu|] f_t^\gamma(d\mu).\end{aligned}$$

Since $|\mu' - \mu| \le \gamma$ we always obtain

$$\frac{1}{\gamma}\frac{d}{dt}\int_0^1 \phi(\mu) f_t^\gamma(d\mu) \le 1.$$

Considering the time scale $\tau = \gamma t$ and letting $f_\tau^\gamma := f_t^\gamma$ with a slight abuse of notation, we obtain

$$\frac{d}{d\tau}\int_0^1 \phi(\mu) f_\tau^\gamma(d\mu) \le 1.$$

Integrating from between τ and τ', we deduce

$$\int_0^1 \phi(\mu) f_{\tau'}^\gamma(d\mu) - \int_0^1 \phi(\mu) f_\tau^\gamma(d\mu) \le |\tau' - \tau|.$$

Taking the supremum over all 1-Lipschitz function ϕ, we deduce

$$W_1(f_{\tau'}^\gamma, f_\tau^\gamma) \le |\tau' - \tau| \tag{4.5}$$

where W_1 is the Wasserstein distance over $P([0,1])$ defined as

$$W_1(f,g) = \sup_{\phi} \int \phi(\mu)\,(f-g)(d\mu) \qquad f,g \in P([0,1]).$$

It is well known that the convergence with respect to this distance is the weak convergence. We refer to the book [38] for more details concerning Wasserstein distances. According to (4.5), the sequence of functions $f^\gamma : [0,T] \to (P(]0,1]), W_1)$ is uniformly equicontinuous. Since $P([0,1])$ is moreover compact for the weak convergence by Prokhorov's theorem, we can apply the Arzelà-Ascoli theorem to obtain a converging subsequence.

4.5 Symmetry of the solution under $(p,\mu) \to (1-p, 1-\mu)$: proof of Prop. 2.

Let f_t satisfying (3.12) for a proportion p of winners, namely

$$\begin{aligned} \frac{d}{dt}(f_t,\phi) = f_t(\{0\})(1-P_\sigma[f_t](0))\phi'(0) - f_t(\{1\})P_\sigma[f_t](1)\phi'(1) \\ + (f_t, (2P_\sigma[f_t]-1)\phi'). \end{aligned} \tag{4.6}$$

Define the measure g_t "flipping" f_t around $\mu = \frac{1}{2}$. Formally,

$$\int \phi(\mu) g_t(d\mu) = \int \phi(1-\mu) f_t(d\mu)$$

for any function ϕ. Taking the time derivative using (4.6) we obtain

$$\begin{aligned} &\frac{d}{dt}\int \phi(\mu)g_t(d\mu) = \frac{d}{dt}\int \phi(1-\mu)f_t(d\mu) \\ &= -f_t(\{0\})(1-P_\sigma[f_t](0))\phi'(1) + f_t(\{1\})P_\sigma[f_t](1)\phi'(0) \\ &\qquad - \int (2P_\sigma[f_t](\mu)-1)\phi'(1-\mu)f_t(d\mu). \end{aligned}$$

Notice that $f_t(\{0\}) = f_t(\{1\})$, $f_t(\{1\}) = f_t(\{0\})$, and $P_\sigma[g_t](1-\mu) = 1 - P_\sigma[f_t](\mu)$ (where the probabilities $P_\sigma[f_t]$ are with p and the $P_\sigma[g_t]$ are with $1-p$). Then

$$\begin{aligned} \frac{d}{dt}\int \phi(\mu)g_t(d\mu) = -g_t(\{1\})P_\sigma[g_t](1))\phi'(1) + g_t(\{0\})(1-P_\sigma[g_t](0))\phi'(0) \\ + \int (2P_\sigma[g_t](1-\mu)) - 1)\phi'(1-\mu)f_t(d\mu). \end{aligned}$$

We conclude noticing the last integral is $\int (2P_\sigma[g_t](\mu)) - 1)\phi'(\mu)g_t(d\mu)$.

4.6 Explicit solutions to the continuous equations.

4.6.1 Proof of Prop. 3.

Looking for a solution $f_t = \delta_{\mu(t)}$ with $\mu(t) \in (0,1)$, we have

$$\phi'(\mu(t))\mu'(t) = \frac{d}{dt}(f_t, \phi) = (f_t, (2P_0[f_t] - 1)\phi') = (2P_0[f_t](\mu(t)) - 1)\phi'(\mu(t))$$

i.e.

$$\mu'(t) = 2P_0[\delta_{\mu(t)}](\mu(t)) - 1 = 2p - 1.$$

Thus $f_t = \delta_{\mu(t)}$, $\mu(t) = (2p-1)t + \mu(0)$. This computations hold until $\mu(t) = 0$ is $p < \frac{1}{2}$ or $\mu(t) = 1$ if $p > \frac{1}{2}$.

4.6.2 Proof of Prop. 4.

The proof is a direct verification that f_t solves (3.12) with $\sigma = 0$. First notice that for any t, $F_t^{-1}(p) = a(t)$ so that

$$P_0[f_t](\mu) = \begin{cases} 1 & \text{if } \mu < a(t), \\ 0 & \text{if } \mu > a(t), \\ \frac{p - f_t([0,a(t))}{f_t(\{a(t)\})} \text{if} & \mu = a(t). \end{cases}$$

For $0 \le t \le p$, $(f_t, \phi) = \int_t^{1-t} \phi + 2t\phi(p)$ so that

$$\frac{d}{dt}(f_t, \phi) = -\phi(1-t) - \phi(t) + 2\phi(p).$$

On the other hand, notice that $P_0[f_t](p) = \frac{1}{2}$ so that the right hand side of (3.12) is

$$(f_t, (2P_0[f_t] - 1)\phi') = \int_t^p \phi' - \int_o^{1-t} \phi' = \frac{d}{dt}(f_t, \phi).$$

This computation holds until there is no more mass on at least one side of $\delta_{a(t)}$. When $p \le \frac{1}{2}$ (resp. $p \ge \frac{1}{2}$), this holds when $1_{[t,p]}$ (resp. $1_{[p,1-t]}$) collapses to δ_p which occurs at time $t = p$ (resp. $t = 1-p$).

Let us examine first the case $p \le \frac{1}{2}$. For $t \ge p$ we look for a solution of the form $f_t = 1_{[a(t),1-t]} + (t + a(t))\delta_{a(t)}$. Then $(f_t, \phi) = \int_{a(t)}^{1-t} \phi + (t + a(t))\phi(a(t))$ so that

$$\frac{d}{dt}(f_t, \phi) = \phi(a(t)) - \phi(1-t) + (t + a(t))\phi'(a(t))a'(t).$$

On the other hand, with $P_0[f_t](a(t)) = \frac{p}{t+a(t)}$, the right hand side of (3.12) is

$$(f_t, (2P_0[f_t] - 1)\phi') = -\int_{a(t)}^{1-t} \phi' + (t + a(t))\phi'(a(t))(\frac{2p}{t + a(t)} - 1).$$

Equating these two expressions, we see we must choose a satisfying $a'(t) = \frac{2p}{t+a(t)} - 1$, $t > p$, with $a(p) = p$. Solving we obtain $a(t) = 2\sqrt{pt} - t$. This computation holds until $1_{[a(t),1-t]}$ collapses to $\delta_{a(t)}$ i.e. $1-t=a(t) \Leftrightarrow t=\frac{1}{4p}$, or $a(t)=0 \Leftrightarrow t=4p$, whatever occurs first. If $\frac{1}{4}<p<\frac{1}{2}$, then $1_{[a(t),1-t]}$ collapses to $\delta_{a(t)}$ before reaching 0. Then from $t \geq \frac{1}{4p}$, f_t is a Dirac mass $\delta_{a(t)}$, and we can use Prop. 3 i.e., $a'(t)=2p-1$. This holds until $a(t)=0$ which occurs at $t=\frac{1}{2p(1-2p)}$.

If $p \geq \frac{1}{2}$, the result follows from the case $p \leq \frac{1}{2}$ and Prop. 2.

4.6.3 Linear stability analysis of $(1-p)\delta_0+p\delta_\gamma$.

We provide some intuition concerning the stability of the approximate stationary state of $f=(1-p)\delta_)+p\delta_\gamma$. Consider a perturbation $\tilde{f}_0=f+h_0$ and denote $\tilde{f}_t$ the corresponding solution. We write $\tilde{f}_t=f+h_t$, $h_t=f+a(t)\delta_0+b(t)\delta_\gamma+g_t$ where $g_t(\{0\})=g_t(\{\gamma\})=0$. We suppose that h_0 is small in the sense that $|a(0)|,|b(0)| \ll 1$ and $\int h_0 \ll 1$. We take $a(0)$ small enough so that $\tilde{f}_0(\{0\})=1-p+a(0)>p$. Then $\tilde{f}_t(\{0\})>p$ up to some time T. For $t<T$, we then have $P[\tilde{f}_t](0)=p/\tilde{f}_t(\{0\})$ and $P[\tilde{f}_t](\mu)=0$ for $\mu>0$, so that

$$\begin{aligned}\frac{d}{dt}(\tilde{f}_t,\phi) &= \tilde{f}_t(\{0\})P_0[\tilde{f}_t](0))\phi'(0)+\tilde{f}_t(\{1\})(P_0[f_t](1)-1)\phi'(1)\\ &\quad +(\tilde{f}_t-\tilde{f}_t(\{0\})-\tilde{f}_t(\{1\}),(2P_0[\tilde{f}_t]-1)\phi')\\ &= p(\phi'(0)-\phi'(\gamma))-b(t)\phi'(\gamma)-(g_t,\phi')\end{aligned}$$

Since $(\phi'(0)-\phi'(\gamma)=O(\gamma)$, neglecting terms of order γ, we obtain

$$a'(t)\phi(0)+b'(t)\phi(\gamma)+\frac{d}{dt}(g_t,\phi)=-b(t)\phi'(\gamma)-(g_t,\phi').$$

Thus from $\frac{d}{dt}(g_t,\phi)=-(g_t,\phi')$, which is the weak form of the transport equation $\partial_t g_t-\partial_\mu g_t=0$, we see that g_t is transported toward the left. We can thus reasonably assume that after some time, all the mass in g_t collapsed to 0 so that h_t is approximately like $h_t=a(t)\delta_0+b(t)\delta_\gamma$. Since $\int h_t=0$, we have $b(t)=-a(t)$. We then obtain

$$a'(t)\phi(0)-a'(t)\phi(\gamma)=a(t)\phi'(\gamma)$$

Writing $\phi(\gamma)=\phi(0)+O(\gamma)$, $\phi'(\gamma)=\phi'(0)+O(\gamma)$ and neglecting terms of order γ, we obtain $a(t)\phi'(0)=0$ for any ϕ, so that $a=0$. We thus obtain $h_t \to 0$.

4.7 Evolution of the mean μ value: proof of (3.14).

The mean value $m(t)=\int \mu\, f_t(d\mu)$ solves the equation

$$m'(t)=-f_t(\{1\})P_0[f_t](1)+f_t(\{0\})(1-P_0[f_t](0))+(f_t,(2P_0[f_t]-1)). \quad (4.7)$$

To simplify the right-hand side we distinguish three cases: $F_t^{-1}(p)=0$, 1 or $F_t^{-1}(p)\in(0,1)$.

If $F_t^{-1}(p) = 1$ then $P[f_t](\mu) = 1_{\mu<1} + \frac{p - f_t([0,1))}{f_t(\{1\})} 1_{\mu=1}$. Then (4.7) becomes

$$m'(t) = f_t([0,1)) - p + f_t(\{1\})(2\frac{p - f_t([0,1)}{f_t(\{1\})} - 1) + f_t([0,1)) = p - f_t(\{1\}).$$

If $F_t^{-1}(p) = 0$ then $P[f_t](\mu) = \frac{p}{f_t(\{0\})} 1_{\mu=0}$. Then (4.7) becomes

$$m'(t) = f_t(\{0\}) \frac{p}{f_t(\{0\})} - f_t((0,1]) = p - (1 - f_t(\{0\}).)$$

Eventually if $F_t^{-1} \in (0,1)$ then $P[f_t](\mu) = 1_{\mu<X_t(p)} + \frac{p - f_t([0,X_t(p)))}{f_t(\{X_t(p)\})} 1_{\mu=X_t(p)}$ so that

$$m'(t) = (f_t, (2P_0[f_t] - 1)) = 2(f_t, P_0[f_t]) - 1$$

with $(f_t, P_0[f_t]) = f_t([0, X_t(p)) + f_t(\{X_t(p)\}) \frac{p - f_t([0,X_t(p)))}{f_t(\{X_t(p)\})} = p$.